New Approach to Inviscid Flow/ Boundary-Layer Matching

G. W. Brune,* P. E. Rubbert,† and T. C. Nark Jr.‡
Boeing Commercial Airplane Company, Seattle, Wash.

Introduction

MANY problems in fluid dynamics require a matching of flow solutions that are calculated in adjacent regions governed by different types of equations. A classical problem of this kind is the calculation of unseparated flow about an airfoil, requiring the matching of inviscid flow and boundary-layer solutions. In the latter case, the governing equations are generally nonlinear and their solution is further complicated by the fact that surface velocity and boundary-layer displacement thickness must simultaneously satisfy the inviscid flow and boundary-layer boundary value problems. The process that determines the particular combination of surface velocity and boundary-layer displacement thickness satisfying simultaneously the inviscid flow and boundary-layer problems is termed "matching" in this Note. In other words, matching is defined here as the simultaneous solution of linear or nonlinear boundary value problems coupled through their boundary conditions. In that sense, matching is distinguished from the matching applied in the context of perturbation theory.[1,2] The latter provides the pertinent flow equations and matching conditions, but does not, in general, give a solution of the resulting coupled boundary value problems.

The classical approach to the calculation of the flow about an airfoil, including both inviscid and boundary-layer domains, is an iterative process that usually begins by calculating the inviscid flow without any representation of viscous flow effects. This provides an initial estimate of the surface velocity, which is then used as the basis for a boundary-layer calculation. The latter provides a first estimate of the boundary-layer displacement thickness. It is well known[3] that the effect of the boundary-layer on the inviscid flow can be approximately taken into account by adding that displacement thickness to the airfoil surface geometry and repeating the inviscid flow calculation. The iteration is frequently terminated at this point, and the results so obtained do provide a useful first-order correction for flows that are only weakly interactive. However, in many flow problems the inviscid flow/boundary-layer interaction is not weak, and it is frequently desired to continue the iteration to seek a fully converged solution. The conventional approach of recalculating the inviscid and boundary-layer flows in successive fashion rarely converges without some assistance, and when it does the convergence is relatively slow. The present Note shows why the conventional approach so frequently fails and demonstrates a new method of matching that seems to be clearly superior.

Overview of Matching Problem

Consider the problem of calculating unseparated flow about an airfoil. A solution of the outer inviscid flow problem depends on boundary conditions dictated by airfoil geometry and boundary-layer displacement thickness δ_p^*. The inviscid solution provides a velocity U_p at the edge of the boundary layer. The relation between U_p and δ_p^* can be written as

$$U_p = U_p(\delta_p^*) \tag{1}$$

where the subscript P denotes quantities in the domain of the inviscid (potential) flow boundary value problem.

A solution of the boundary-layer equations and the subsequent determination of a corresponding displacement thickness are dependent on a surface velocity U_B, which can be viewed as a boundary condition for the boundary-layer equations. A solution for the displacement thickness δ_B^* as provided by the boundary-layer equations can be written as

$$\delta_B^* = \delta_B^* \ (U_B) \tag{2}$$

where the subscript B denotes quantities in the domain of the boundary-layer equations.

The matching process seeks to establish particular values of U and δ^* satisfying the relations

$$\delta_p^* = \delta_B^* \tag{3}$$

and

$$U_p = U_B \tag{4}$$

while simultaneously satisfying Eqs. (1) and (2).

New Matching Procedure

The relationships implicit in Eqs. (1) and (2) can be viewed conceptually in simplified form as representing curves or traces in $U-\delta^*$ space, such as depicted in Fig. 1. In $U-\delta^*$ space, such as depicted in Fig. 1. In $U-\delta^*$ space, it is apparent that the matching process becomes one of determining the point of intersection of the two respective curves, one curve representing the relationship between surface velocity and displacement thickness as provided by the inviscid flow equations, and the other curve representing the relationship between displacement thickness and surface velocity as provided by the boundary-layer equations.

The new approach to inviscid flow/boundary-layer matching is based on the observation that small perturbations to nonlinear flows turn out to be governed by linear equations and are thereby amenable to linear analysis techniques, even though the basic flow is nonlinear.[4,5] Surface velocity $U(x)$ and displacement thickness $\delta^*(x)$ are the variables that are perturbed in the present problem. For the inviscid flow problem, a change in the surface velocity, $\Delta\delta_p^*(x)$, is linearly related to a corresponding change in the surface velocity, $\Delta U_p(x)$. Consequently, ΔU_p and $\Delta\delta_p^*$ can be discretized in a numerical solution procedure and the relationship between them expressed in a linear influence coefficient representation of the form

$$\{\Delta U_{p_i}\} = [P_{ij}] \ \{\Delta\delta_{P_j}^*\} \tag{5}$$

where the $[P_{ij}]$ are the inviscid flow influence coefficients.

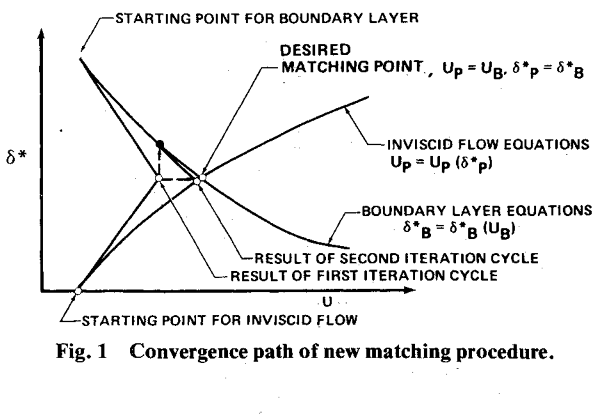

Fig. 1 Convergence path of new matching procedure.

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*Senior Engineer. Member AIAA.

†Supervisor, Aerodynamic Analysis Group. Member AIAA.

‡Unit Chief, Aerodynamics Research Group. Member AIAA.

Similarly, the perturbed boundary-layer equations can be reduced by discretization to a linear influence coefficient formulation relating small changes in displacement thickness to small changes in surface velocity in a linear manner; i.e.,

$$\{\Delta\delta^*_{B_i}\} = [B_{ij}]\ \{\Delta U_{B_j}\} \tag{6}$$

where $[B_{ij}]$ is a set of boundary-layer influence coefficients.

The new matching procedure is an iteration scheme, whose convergence path is illustrated in Fig. 1. It proceeds as follows:

Step 1: An initial guess is selected for δ^*_p, the value of displacement thickness in the potential flow domain. (This can be chosen to be zero.)

Step 2: The inviscid flow boundary value problem Eq. (1) is solved for U_p, the surface velocity in the potential flow domain. The result of steps 1 and 2 is termed "starting point for the inviscid flow" in Fig. 1.

Step 3: An arbitrary distribution of U_B, the surface velocity in the boundary-layer domain, is selected. The choice $U_B = U_p$ is convenient but not necessary.

Step 4: The boundary-layer equations (2) are solved to determine the value of δ^*_B corresponding to U_B. The result of steps 3 and 4 is shown in Fig. 1 as the "starting point for the boundary layer."

Step 5: A first approximation to a matched solution is obtained by setting

$$\{\delta^*_p\} + \{\Delta\delta^*_p\} = \{\delta^*_B\} + \{\Delta\delta^*_B\} \tag{7}$$

$$\{U_p\} + \{\Delta U_p\} = \{U_B\} + \{\Delta U_B\} \tag{8}$$

and solving simultaneously Eqs. (5-8) for the perturbation values ΔU_p, ΔU_B, $\Delta\delta^*_p$, $\Delta\delta^*_B$.

The result of the first iteration cycle, consisting of steps 1-5 is

$$\{U_i\} = \{U_{pi}\} + \{\Delta U_{pi}\} \tag{9}$$

$$\{\delta^*_i\} = \{\delta^*_{Bi}\} + \{\Delta\delta^*_{Bi}\} \tag{10}$$

which is shown in Fig. 1 as the point of intersection of straight line tangents to the inviscid flow and boundary-layer solution curves emanating from the starting points. The perturbation equations (5) and (6) define conceptually these tangent lines. In essence, the process is seen to be equivalent to approximating the solution curves in $U-\delta^*$ space by straight lines locally tangent to the starting points and solving for the point at which these tangent lines intersect. The deviation of this intersection point from the exact matching point is directly related to the size of the second-order terms that were neglected in the linear approximation.

The process is recycled by using the updated values of δ^* and U at the intersection point of the tangent lines to establish new starting points in the inviscid flow and boundary-layer domains. Step 1 and 3 are replaced by setting $\delta^*_p = \delta^*$ and $U_B = U$, respectively. In the Fig. 1 schematic, the process is essentially converged in two iterations.

Conventional Matching Procedure

The conventional approach of recalculating the inviscid and boundary-layer flows in successive fashion is depicted conceptually in Fig. 2 for two cases: convergent and divergent. The cycle path is seen to be a spiral, which has poor convergence characteristics. It is readily apparent how subtle features in the shapes of the conceptual curves representing the relations between U and δ^* as given by the inviscid flow and boundary-layer boundary value problems can lead to widely differing convergence properties. Figure 2a illustrates a case of a convergent spiral, and Fig. 2b a case of a divergent spiral. Numerical examples discussed below support this view of conventional matching as being, at best, a marginal and slowly converging approach and, at worst, a divergent process.

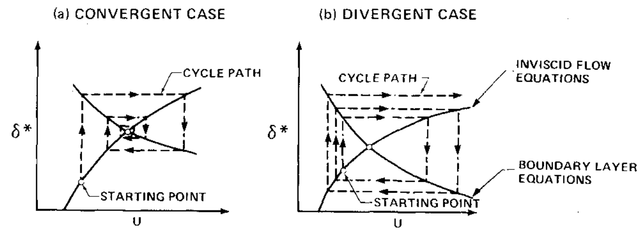

Fig. 2 Cycle paths using conventional approach.

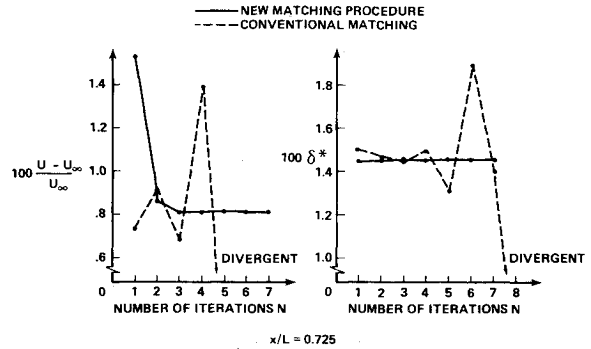

Fig. 3 Convergence properties for flat plate problem, $Re_L = 10^4$.

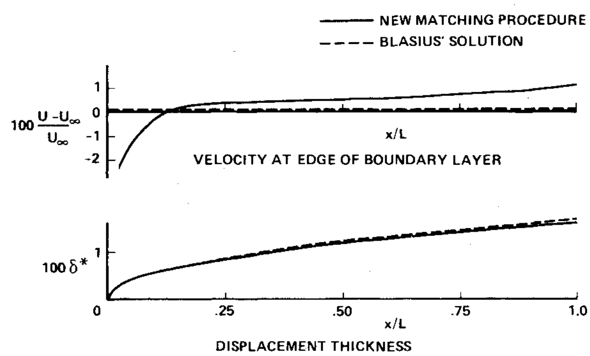

Fig. 4 Complete flat plate solution, $Re_L = 10^4$.

Numerical Convergence Study

The familiar problem of laminar incompressible flow over a flat plate is used as an example to demonstrate and compare the properties of the new and the conventional matching procedures. For simplicity, boundary-layer growth was represented by the quadrature formula of Walz.[6] The displacement effect of the boundary layer on the outer potential flow was represented by a source distribution along the flat plate surface. Details of the potential flow and boundary-layer formulations and of the numerical scheme employed for their discretization are contained in Ref. 7.

Convergence properties were studied at Reynolds numbers of 10^4 and 10^5 based on plate length. The behavior at the lower Reynolds number of 10^4 is shown in Fig. 3. Plotted is the behavior of surface velocity U and displacement thickness δ^* as a function of iteration number for a point at 72.5% plate length. The values of U and δ^* associated with the conventional matching procedure were taken from the boundary-layer solution $\delta^*_B = \delta^*_B(U_B)$ at each iteration step. The values of δ^* and U from the new matching technique were the values corresponding conceptually to the point of intersection of the straight lines tangential to the boundary layer and inviscid flow solution curves as shown in Fig. 1. In this case there is a strong interaction between the thick boundary layer and the inviscid flow. The new procedure converged after three iteration steps, whereas the conventional matching technique diverged. The complete converged flat plate solution of surface velocity and displacement thickness obtained via the new matching method is shown in Fig. 4 for a Reynolds number of

10^4. Matched solutions were obtained at $R_{e_L} = 10^5$ with both matching procedures. Convergence was found monotonic with the new matching procedure and oscillatory (for the surface velocity) with the conventional matching procedure.[7]

Conclusions

A simple, flexible, and powerful method of matching solutions of different flow regions along a common boundary has been presented. Successful matching and rapid convergence to an accurately matched solution has been demonstrated for the problem of laminar flow over a flat plate. The method is obviously applicable to a much broader class of flow problems and their mathematical representations. It is apparent that the method can be applied to boundary-layer representations formulated in terms of integral relations or finite differences for either laminar or turbulent flows with appropriate turbulence modeling. Boundary layers, in turn, can be matched to any of several standard inviscid flow representations including numerical formulations such as linear influence coefficient methods, loading functions, or finite difference methods. Two examples of this kind, namely transonic small disturbance theory and finite difference representations of laminar boundary layers, are briefly discussed in Ref. 7.

Yet to be explored are the refinements that may be required when singular points occur in the matching, such as may appear as nonanalytic behavior in $dU/d\delta^*$ and/or $d\delta^*/dU$ in the vicinity of separation, shock/boundary-layer interaction, etc. In such cases it may be desirable to consider an additional imbedded region containing a mathematical description of the local phenomena and matched to the surrounding flow. An example of such a treatment for a problem involving separated flow is reported in Ref. 8.

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Vibration and Stability of Anti-Adjoint Elastic Systems

C. Sundararajan*
The Lummus Company Canada Limited, Willowdale, Toronto, Canada

STABILITY investigations of elastic systems lead to the eigenvalue analysis of homogeneous boundary-value problems. If the system is conservative in nature, the problem is self-adjoint, and if it is nonconservative in nature, the problem is nonself-adjoint. In this paper a special class of nonself-adjoint systems, namely, "anti-adjoint systems," is defined and the behavior of such systems studied.

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*Engineer.

Definition

Consider an undamped, continuous, linearly elastic system occupying a domain V, inside a closed boundary S. Let the equation of motion be

$$\mu\, \partial^2 w/\partial t^2 + K[w] + PF[w] = 0 \tag{1}$$

where $w = w(x_j, t)$ = predominant deflection from the equilibrium position; x_j = spatial coordinates, t = time; $\mu = \mu(x_j)$ = mass density; P = force parameter; K = a self-adjoint operator in x_j, where $K[w]$ represents the stiffness; and F = a self-adjoint or nonself-adjoint operator in x_j, where $PF[w]$ represents the forces. Let the boundary conditions be

$$B[w] = 0 \quad \text{on } S \tag{2}$$

where B is a self-adjoint or nonself-adjoint operator in x_j.

Let the adjoint system be represented by the equation of motion

$$\mu\, \partial^2 u/\partial t^2 + K[u] + PF^*[u] = 0 \tag{3}$$

and the boundary conditions

$$B^*[u] = 0 \quad \text{on } S \tag{4}$$

where $u = u(x_j, t)$ = predominant deflection of the adjoint system, similar to w, and F^* and B^* are operators similar to F and B respectively.

If the system is self-adjoint

$$F^* \equiv F \quad \text{and} \quad B^* \equiv B \tag{5}$$

If at least one of these equalities does not hold, the system is nonself-adjoint. If

$$F^* \equiv -F \quad \text{and} \quad B^* \equiv B \tag{6}$$

such systems may be called "anti-adjoint systems."

Buckling Instability

Nonself-adjoint (nonconservative) systems can have two types of instability mechanisms—divergence (buckling) and flutter (oscillatory instability). Buckling is a static phenomenon, and at buckling the total potential energy is zero; i.e.,

$$\int_V K[w]\, w\, dV + P \int_V F[w]\, w\, dV = 0 \tag{7}$$

It is shown in the Appendix that for an anti-adjoint system

$$\int_V F[w]\, w\, dV = 0 \tag{8}$$

So, at buckling

$$\int_V K[w]\, w\, dV = 0 \tag{9}$$

However, since K represents the stiffness of the system, the Eq. (9) integral is positive definite and so the equation holds only for trivial solutions of w. Hence the anti-adjoint system can never buckle.

Reversal of Forces

Substitution of Eq. (6) in Eqs. (3) and (4) shows that the adjoint system of an anti-adjoint system can be obtained by